Maxwell Equations and Special Relativity in Accelerators

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Motivation

- Need to know how particles will move in the presence of electric and magnetic fields. Present a basic review of classical physics*
  - Equations of Motion
  - Calculations of the Fields
  - Special Relativity
- Give a couple examples

** Will ignore quantum mechanical effects for now*

References

- Feynman Lectures by R. Feynman, R. Leighton, and M Sands
- Introduction to Electrodynamics by D. Griffith
- Spacetime Physics by E. Taylor and J. Archibald
- Particle Accelerator Physics, Basic Principles and Linear Beam Dynamics by H. Wiedemann

Equations of Motion

Newton's Law's of Motion

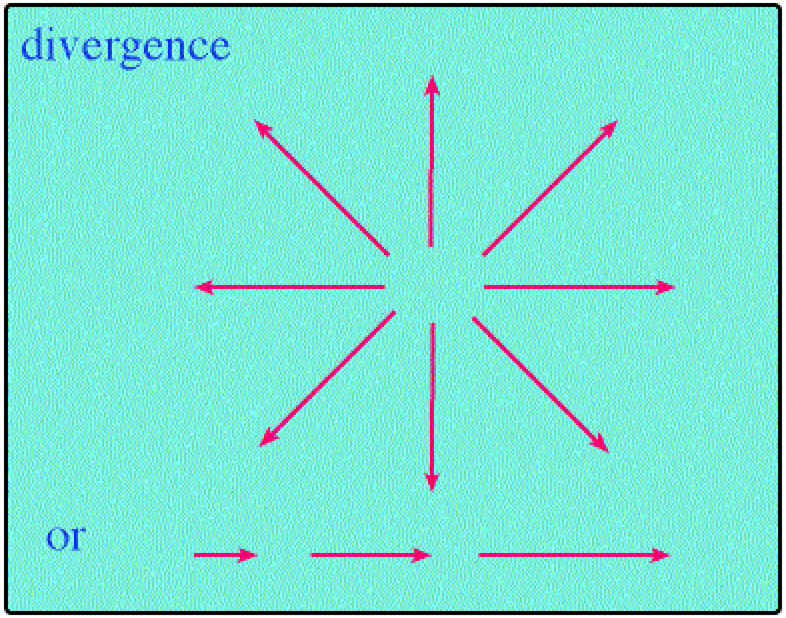
$$\vec{F} = m\vec{a}$$

Lorentz Force Equation – Force on a charged particle traveling with velocity, v, in the presence of an electric, *E*, or magnetic, *B*, field

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

To determine the particle motion one needs to know the electric and magnetic fields – Maxwell's Equations

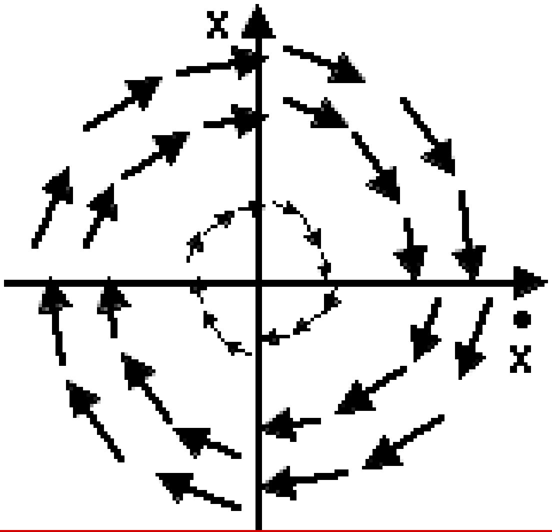
Fundamental Theorems



I. Divergence Theorem

$$\int (\nabla \cdot A) d\tau = \oint A \cdot da$$

II. Curl Theorem



$$\int (\nabla \times A) da = \oint A \cdot dI$$

Vectorial Algebra

$$\nabla\cdot\left(\nabla\times\overline{F}\right)=0 \quad \forall\overline{F} \qquad\qquad \nabla\times\left(\nabla\times\overline{F}\right)=\nabla\left(\nabla\cdot\overline{F}\right)-\nabla^2\overline{F} \quad \forall\overline{F}$$

$$\nabla\times\left(\nabla u\right)=0 \qquad \forall u$$

or

$$\nabla\times\overline{F}=0 \quad \Leftrightarrow \quad \overline{F}=\nabla u$$

(***F*** is conservative if curl ***F*** is zero)

Volume Integral

$$\int_S \overline{F}\cdot\overline{n}\; dS=\int_V \nabla\cdot\overline{F}\; dV$$

Divergence Theorem

Surface Integral (Flux)

$$\oint_l \overline{F}\cdot d\overline{l}=\int_S \left(\nabla\times\overline{F}\right)\cdot\overline{n}\, dS$$

Curl Theorem (Stoke's Theorem)

Line Integral (Circuitation)

Maxwell's Equations

I. Gauss' Law

(Flux of *E* through a closed surface) = (Charge inside/ε_0)

$$\nabla \cdot E = \frac{\rho}{\varepsilon_o}$$

II. (No Name)

(Flux of *B* through a closed surface) = 0

$$\nabla \cdot B = 0$$

permittivity of free space

$$\varepsilon_o = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{Nm}^2$$

Maxwell's Equations

III. Faraday's Law

(Line Integral of *E* around a loop) =
-d/dt(Flux of B through the loop)

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

IV. (Ampere's Law modified by Maxwell)

(Integral of B around a loop) = (Current through the loop)/ε_0
+d/dt(Flux of E through the loop)

$$\nabla \times B = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

permeability of free space

$$\mu_o = 4\pi \times 10^{-7}\, N / \mathrm{A}^2$$

Maxwell's Equations and Light

Equation of a wave in three dimensions is

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial^2 t} \quad \text{where } v \text{ is the velocity of the wave}$$

In free space combining Gauss' Law and Faraday's Law

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial^2 t}$$

In free space combining Ampere's Law and the last Maxwell Equations

$$\nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial^2 t}$$

From Maxwell Equations to Wave Equation

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\mathbf{E}$$

$\mathbf{E} = \mathbf{B} = 0$ is a solution, but there might be other solutions as well. Let us employ a useful identity from vector calculus.

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Where $\mathbf{A}$ can be any vector function. Taking the curl of the curl equations and applying the identity, we get the following.

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial^2 t}\mathbf{E}$$

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial^2 t}\mathbf{B}$$

Maxwell's Equations and Light

Equation of a wave in three dimensions is

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial^2 t} \text{ where } v \text{ is the velocity of the wave}$$

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3.00 \times 10^8 \, m/s$$

Maxwell's equations imply that empty space supports the propagation of electromagnetic waves traveling at the speed of light

Perhaps Light is an electromagnetic wave

"We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena" - Maxwell

Maxwell's Equations in Integral Form

$$\int \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int \rho dV$$

$$\int \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{d\vec{B}}{dt} \cdot d\vec{S}$$

$$\oint \vec{B} \cdot d\vec{l} = -\mu_0 \int \vec{j} \cdot d\vec{S} + \varepsilon_0 \mu_0 \int \frac{d\vec{E}}{dt} \cdot d\vec{S}$$

Special Relativity

1. **The principle of relativity**. The laws of physics apply in all inertial reference systems.

2. **The universal Speed of light.** The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

Lorentz Factor, γ

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where *v* is the velocity of the particle and *c* is the velocity of light

$$\beta = \frac{v}{c}$$

$$\text{particle momentum}: p = m\gamma v$$

where *m* is the rest mass of the particle

$$\vec{F} = \frac{dp}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Energy and Momentum

Rest Energy, E_o : $E_o = mc^2$

Total Energy, E : $E = m\gamma c^2$

Momentum, p : $p = m\gamma v$

$$E^2 = E_0^2 + p^2 c^2$$

Lorentz Particle collisions

- Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E.

$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Mass Frame (CMF): Velocities are equal and opposite, total energy is E_{cm}.

$$\mathbf{P_1} = (E_{\mathrm{cm}}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\mathrm{cm}}/(2c), -\mathbf{p})$$

- The quantity $(\mathbf{P_1} + \mathbf{P_2})^2$ is invariant.
- In the **CMF**, we have $(\mathbf{P_1} + \mathbf{P_2})^2 = E_{\mathrm{cm}}^2/c^2$.
- In general $(\mathbf{P_1} + \mathbf{P_2})^2 = \mathbf{P_1}^2 + \mathbf{P_2}^2 + 2\mathbf{P_1}\cdot\mathbf{P_2} = 2m_0^2c^2 + 2\mathbf{P_1}\cdot\mathbf{P_2}$
- In the **LF**, we have $\mathbf{P_1}\cdot\mathbf{P_2} = E_1 m_0$ and $(\mathbf{P_1} + \mathbf{P_2})^2 = 2m_0 E$.
- And finally $E_{\mathrm{cm}}^2 = 2m_0c^2E$

Lorentz Transformation

Two inertial frames moving with respect to each other with velocity, *v*

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

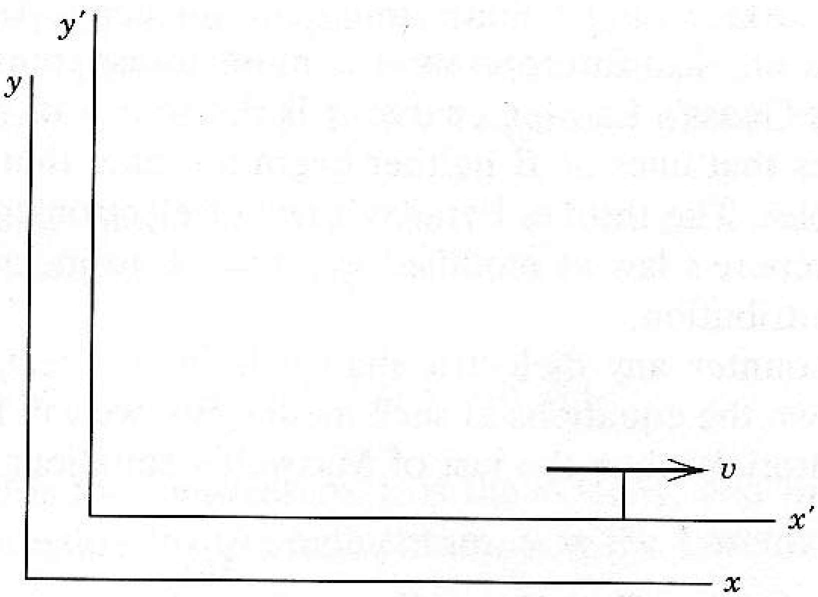


Figure 1.4. Inertial reference frames moving with respect to one another with relative speed *v*.

Time Dilation and Lorentz Contraction

Two celebrated consequences of the transformation are
Time dilation and Lorentz contraction

Time dilation. A clock in the primed frame located at $x = vt$ will show a time dilation, $t' = 1/\gamma$

Lorentz contraction. An object in the primed frame with length L' along the x' axis and is at rest in the primed frame will be of length $L = L'/\gamma$ in the unprimed frame

Lorentz Transformation of Electric and Magnetic Fields

$$E'_x = E_x$$
$$E'_y = \gamma\left(E_y - v B_z\right)$$
$$E'_z = \gamma\left(E_z + v B_y\right)$$

$$B'_x = B_x$$
$$B'_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right)$$
$$B'_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

Thanks

Wish to thank Y. Papaphilippou and N.Catalan-Lasheras for sharing the tranparencies that they used in the USPAS, Cornell University, Ithaca, NY 20th June – 1st July 2005

L2 Possible Homework

Problem 1.

Protons are accelerated to a kinetic energy of 200 MeV at the end of the Fermilab Alvarez linear accelerator. Calculate their total energy, their momentum and their velocity in units of the velocity of light.

Problem 2.

A charge pion has a rest energy of 139.568 MeV and a mean life time of τ = 26.029 nsec in its rest frame. What are the pion life times, if accelerated to a kinetic energy of 20 MeV? And 100 MeV? A pion beam decays exponentially like $e^{-\tau/t}$. At what distance from the source will the pion beam intensity have fallen to 50%, if the kinetic energy is 20 MeV? Or 100 MeV?

Problem 3.

A positron beam accelerated to 50 GeV in the linac hits a fixed hydrogen target. What is the available energy from a collision with a target electron assumed to be at rest? Compare this available energy with that obtained in a linear collider where electrons and positrons from two similar linacs collide head on at the same energy.

Rest energy of an electron = 0.511 MeV

Rest energy of a proton = 936 MeV

L2 Possible Homework

- Show that a function satisfying $f(x,t)=f(x-vt)$ automatically satisfies the wave equation.
- A muon has a rest mass of 105.7MeV and a lifetime at rest of 2.2e-6 s.
  - Consider a muon traveling at 0.9c with respect to the lab frame. What is its lifetime? How far does the muon travel? How does this compare to the distance it would travel if there were no time dilation?
  - Consider a muon accelerated to 1GeV.
  - What is its velocity? How long does it live?
- For a non-relativistic charge moving in the z direction, calculate the general particle trajectory when subjected to a field $B_x=B_z=0$, and $B_y=\sin(2\pi z/\lambda)$ for $0<z<z_0$, and $B_y=0$ elsewhere.